# Problem Sheet 8

**Problem 1**

Let $\chi\colon (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character and $K \subseteq \mathbb{Q}(\zeta_N)$ the fixed field of its kernel. Prove that $\chi(-1) = 1$ if and only if $K$ is a totally real field. The latter meaning that for every $\nu\colon K \to \mathbb{C}$ one has $\nu(K) \subseteq \mathbb{R}$.

**Problem 2**

(a) Let $a_n, b_n \in \mathbb{C}$ be two sequences of complex numbers and for $m \leq k$, $m \leq m'$ put

$$A_{m,k} = \sum_{n=m}^{k} a_n \quad \text{and} \quad S_{m,m'} = \sum_{n=m}^{m'} a_n b_n.$$

Prove

$$S_{m,m'} = \sum_{n=m}^{m'-1} A_{m,n}(b_n - b_{n+1}) + A_{m,m'} b_{m'}.$$

(b) Let $0 < \alpha < \beta$ real numbers and let $z = x + iy \in \mathbb{C}$, $x, y \in \mathbb{R}$, $x > 0$. Then

$$|e^{-\alpha z} - e^{-\beta z}| \leq \left|\frac{z}{x}\right| (e^{-\alpha x} - e^{-\beta x}).$$

*Hint: Write* $e^{-\alpha z} - e^{-\beta z} = z \int_\alpha^\beta e^{-tz} dt$.

(c) Let $f(s) = \sum\limits_{n=1}^{\infty} \frac{a_n}{n^s}$, $a_n \in \mathbb{C}$, be a Dirichlet series. Prove that if $f$ converges for some $s_0 \in \mathbb{C}$, then $f$ converges locally uniformly on $\{ s \in \mathbb{C} \mid \mathrm{Re}(s) > \mathrm{Re}(s_0) \}$.

**Problem 3**

Let $A$ be a subset of rational primes. If there exists a number $\rho \in [0,1]$ such that

$$\sum_{p \in A} \frac{1}{p^s} \sim \rho \log \frac{1}{s-1} \text{ for } s \text{ real and } s \to 1^+,$$

(i.e. $s$ approaches 1 along the real line from the right), then we say that $A$ has Dirichlet density $\rho$. Compute the Dirichlet density for

$$A_n := \{ p \mid 2 \text{ is an } n\text{th power mod } p \}$$

and $n = 2, 3$.
*Hint for $n = 3$: Consider the Dedekind $\zeta$-function for* both $\mathbb{Q}(\sqrt[3]{2})$ *and its Galois closure.*

**Problem 4**

More intuitive than the Dirichlet density is the so-called *natural density* and we aim to show that the latter is also a former, if exists. Let $A$ be a subset of primes. For $x > 0$, denote by $\pi(x)$ the number of all primes less than $x$, and by $\pi_A(x)$ the number of primes in $A$ less than $x$. The natural density of $A$ is defined as the limit

$$\rho := \lim_{x \to \infty} \frac{\pi_A(x)}{\pi(x)}$$

whenever it exists.

(a) Show that

$$\sum_{p \in A} \frac{1}{p^s} - \rho \sum_p \frac{1}{p^s} = \sum_{n=1}^{\infty} (\pi_A(n) - \rho\pi(n)) \left( \frac{1}{n^s} - \frac{1}{(n+1)^s} \right).$$

*Hint: Use Exercise 2, Part (a)*

(b) For every $\varepsilon > 0$, there exists $N > 0$ such that $|\pi_A(n) - \rho\pi(n)| \leq \varepsilon\pi(n)$ for all $n > N$. Use this to prove that for any $\varepsilon' > 0$, there exists $\delta > 0$ such that

$$\left| \sum_{p \in A} \frac{1}{p^s} - \rho \sum_p \frac{1}{p^s} \right| \leq \varepsilon' \sum_p \frac{1}{p^s}$$

whenever $1 < s < 1 + \delta$. Conclude that $A$ has Dirichlet density $\rho$.